

COEFFICIENT, DISTORTION AND GROWTH INEQUALITIES FOR CERTAIN CLOSE-TO-CONVEX FUNCTIONS

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ABSTRACT. In the present investigation, certain subclasses of close-to-convex functions are investigated. In particular, we obtain an estimate for the Fekete-Szegő functional for functions belonging to the class, distortion, growth estimates and covering theorems.

1. INTRODUCTION

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane \mathbb{C} . Let \mathcal{A} be the class of analytic functions defined on \mathbb{D} and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Let \mathcal{S} be the subclass of \mathcal{A} consisting of univalent functions. Sakaguchi [7] introduced a class of functions called starlike functions with respect to symmetric points; they are the functions $f \in \mathcal{A}$ satisfying the condition

$$\operatorname{Re} \frac{zf'(z)}{f(z) - f(-z)} > 0.$$

These functions are close-to-convex functions. This can be easily seen by showing that the function $(f(z) - f(-z))/2$ is a starlike function in \mathbb{D} . Motivated by the class of starlike functions with respect to symmetric points, Gao and Zhou [2] discussed a class \mathcal{K}_s of close-to-convex univalent functions. A function $f \in \mathcal{K}_s$ if it satisfy the following inequality

$$\operatorname{Re} \left(\frac{z^2 f'(z)}{g(z)g(-z)} \right) < 0 \quad (z \in \mathbb{D})$$

for some function $g \in S^*(1/2)$. The idea here is to replace the average of $f(z)$ and $-f(-z)$ by the corresponding product $-g(z)g(-z)$ and the factor z is included to normalize the expression so that $-z^2 f'(z)/(g(z)g(-z))$ takes the value 1 at $z = 0$. To make the functions univalent, it is further assumed that g is starlike of order $1/2$ so that the function $-g(z)g(-z)/z$ is starlike which in turn implies the close-to-convexity of f . For some recent works on the problem, see [11, 9, 10, 12]. In stead of requiring the quantity $-z^2 f'(z)/(g(z)g(-z))$ to lie in the right-half plane, we can consider more general regions. This could be done via subordination between analytic functions.

Let f and g be analytic in \mathbb{D} . Then f is subordinate to g , written $f \prec g$ or $f(z) \prec g(z)$ ($z \in \mathbb{D}$), if there is an analytic function $w(z)$, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. In particular, if g is univalent in \mathbb{D} , then f is subordinate to g ,

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if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. In terms of subordination, a general class $\mathcal{K}_s(\varphi)$ is introduced in the following definition.

Definition 1. [11] For a function φ with positive real part, the class $\mathcal{K}_s(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying

$$(1.1) \quad -\frac{z^2 f'(z)}{g(z)g(-z)} \prec \varphi(z) \quad (z \in \mathbb{D})$$

for some function $g \in S^*(1/2)$.

This class was introduced by Wang, Gao and Yuan [11]. A special subclass $\mathcal{K}_s(\gamma) := \mathcal{K}_s(\varphi)$ where $\varphi(z) := (1 + (1 - 2\gamma)z)(1 - z)$, $0 \leq \gamma < 1$, was recently investigated by Kowalczyk and Leś-Bomba [5]. They proved the sharp distortion and growth estimates for functions in $\mathcal{K}_s(\gamma)$ as well as some sufficient conditions in terms of the coefficient for function to be in this class $\mathcal{K}_s(\gamma)$.

In the present investigation, we obtain a sharp estimate for the Fekete-Szegő functional for functions belonging to the class $\mathcal{K}_s(\varphi)$. In addition, we also investigate the corresponding problem for the inverse functions for functions belonging to the class $\mathcal{K}_s(\varphi)$. Also distortion, growth estimates as well as covering theorem are derived. Some connection with earlier works are also indicated.

2. FEKETE-SZEGŐ INEQUALITY

In this section, we assume that the function $\varphi(z)$ is an univalent analytic function with positive real part that maps the unit disk \mathbb{D} onto a starlike region which symmetric with respect to real axis and is normalized by $\varphi(0) = 1$ and $\varphi'(0) > 0$. In such case, the function φ has an expansion of the form $\varphi(z) = 1 + B_1 z + B_2 z^2 + \dots$, $B_1 > 0$.

Theorem 2.1 (Fekete-Szegő Inequality). *For a function $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ belonging to the class $\mathcal{K}_s(\varphi)$, the following sharp estimate holds:*

$$|a_3 - \mu a_2^2| \leq 1/3 + \max(B_1/3, |B_2/3 - \mu B_1^2/4|) \quad (\mu \in \mathbb{C}).$$

Proof. Since the function $f \in \mathcal{K}_s(\varphi)$, there is a normalized analytic function $g \in S^*(1/2)$ such that

$$-\frac{z^2 f'(z)}{g(z)g(-z)} \prec \varphi(z).$$

By using the definition of subordination between analytic function, we find a function $w(z)$ analytic in \mathbb{D} , normalized by $w(0) = 0$ satisfying $|w(z)| < 1$ and

$$(2.1) \quad -\frac{z^2 f'(z)}{g(z)g(-z)} = \varphi(w(z)).$$

By writing $w(z) = w_1 z + w_2 z^2 + \dots$, we see that

$$(2.2) \quad \varphi(w(z)) = 1 + B_1 w_1 z + (B_1 w_2 + B_2 w_1^2) z^2 + \dots$$

Also by writing $g(z) = z + g_2 z^2 + g_3 z^3 + \dots$, a calculation shows that

$$-\frac{g(z)g(-z)}{z} = 1 + (2g_3 - g_2^2)z^3 + \dots$$

and therefore

$$-\frac{z}{g(z)g(-z)} = 1 - (2g_3 - g_2^2)z^3 + \dots$$

Using this and the Taylor's expansion for $zf'(z)$, we get

$$(2.3) \quad -\frac{z^2 f'(z)}{g(z)g(-z)} = 1 + 2a_2 z + (3a_3 - 2g_3 + g_2^2)z^2 + \dots$$

Using (2.1), (2.2) and (2.3), we see that

$$\begin{aligned} 2a_2 &= B_1 w_1, \\ 3a_3 &= 2g_3 - g_2^2 + B_1 w_2 + B_2 w_1^2. \end{aligned}$$

This shows that

$$a_3 - \mu a_2^2 = (2/3)(g_3 - g_2^2/2) + (B_1/3)(w_2 + (B_2/B_1 - 3\mu B_1/4)w_1^2).$$

By using the following estimate ([4, inequality 7, p. 10])

$$|w_2 - tw_1^2| \leq \max\{1, |t|\} \quad (t \in \mathbb{C})$$

for an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ which is sharp for the functions $w(z) = z^2$ or $w(z) = z$, the desired result follows upon using the estimate that $|g_3 - g_2^2/2| \leq 1/2$ for analytic function $g(z) = z + g_2 z^2 + g_3 z^3 + \dots$ which is starlike of order $1/2$.

Define the function f_0 by

$$f_0(z) = \int_0^z \frac{\varphi(w)}{1-w^2} dw.$$

The function clearly belongs to the class $\mathcal{K}_s(\varphi)$ with $g(z) = z/(1-z)$. Since

$$\frac{\varphi(w)}{1-w^2} = 1 + B_1 w + (B_2 + 1)w^2 + \dots,$$

we have

$$f_0(z) = z + (B_1/2)z^2 + (1/3 + B_2/3)z^3 + \dots$$

Similarly, define f_1 by

$$f_1(z) = \int_0^z \frac{\varphi(w^2)}{1-w^2} dw.$$

Then

$$f_1(z) = z + (B_1/3 + 1/3)z^3 + \dots$$

The functions f_0 and f_1 show that the results are sharp. \square

Remark 2.1. By setting $\mu = 0$ in Theorem 2.1, we get the sharp estimate for the third coefficient of functions in $\mathcal{K}_s(\varphi)$:

$$|a_3| \leq 1/3 + (B_1/3) \max(1, |B_2|/B_1),$$

while the limiting case $\mu \rightarrow \infty$ gives the sharp estimate $|a_2| \leq B_1/2$. In the special case where $\varphi(z) = (1+z)/(1-z)$, the results reduce to the corresponding one in [2, Theorem 2, p. 125].

Though Xu et al. [12] have given an estimate of $|a_n|$ for all n , their result is not sharp in general. For $n = 2, 3$, our results provide sharp bounds.

It is known that every univalent function f has an inverse f^{-1} , defined by

$$f^{-1}(f(z)) = z, \quad z \in \mathbb{D}$$

and

$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right).$$

Corollary 2.1. *Let $f \in \mathcal{K}_s(\varphi)$. Then the coefficients d_2 and d_3 of the inverse function $f^{-1}(w) = w + d_2w^2 + d_3w^3 + \cdots$ satisfy the inequality*

$$|d_3 - \mu d_2^2| \leq 1/3 + \max(B_1/3, |B_2/3 - (2 - \mu)B_1^2/4|) \quad (\mu \in \mathbb{C}).$$

Proof. A calculation shows that the inverse function f^{-1} has the following Taylor's series expansion:

$$f^{-1}(w) = w + a_2w^2 + (2a_2^2 - a_3)w^3 + \cdots.$$

From this expansion, it follows that $d_2 = a_2$ and $d_3 = 2a_2^2 - a_3$ and hence

$$|d_3 - \mu d_2^2| = |a_3 - (2 - \mu)a_2^2|.$$

Our result follows at once from this identity and Theorem 2.1. \square

3. DISTORTION AND GROWTH THEOREMS

The second coefficient of univalent function plays an important role in the theory of univalent function; for example, this leads to the distortion and growth estimates for univalent functions as well as the rotation theorem. In the next theorem, we derive the distortion and growth estimates for the functions in the class $\mathcal{K}_s(\varphi)$. In particular, if we let $r \rightarrow 1-$ in the growth estimate, it gives the bound $|a_2| \leq B_1/2$ for the second coefficient of functions in $\mathcal{K}_s(\varphi)$.

Theorem 3.1. *Let φ be an analytic univalent functions with positive real part and*

$$\phi(-r) = \min_{|z|=r<1} |\phi(z)|, \quad \phi(r) = \max_{|z|=r<1} |\phi(z)|.$$

If $f \in \mathcal{K}_s(\varphi)$, the following sharp inequalities holds:

$$\begin{aligned} \frac{\varphi(-r)}{1+r^2} &\leq |f'(z)| \leq \frac{\varphi(r)}{1-r^2} \quad (|z| = r < 1), \\ \int_0^r \frac{\varphi(-t)}{1+t^2} dt &\leq |f(z)| \leq \int_0^r \frac{\varphi(t)}{1-t^2} dt \quad (|z| = r < 1). \end{aligned}$$

Proof. Since the function $f \in \mathcal{K}_s(\varphi)$, there is a normalized analytic function $g \in S^*(1/2)$ such that

$$(3.1) \quad -\frac{z^2 f'(z)}{g(z)g(-z)} \prec \varphi(z).$$

Define the function $G : \mathbb{D} \rightarrow \mathbb{C}$ by the equation

$$G(z) := -\frac{g(z)g(-z)}{z}.$$

Then it is clear that G is odd starlike function in \mathbb{D} and therefore

$$\frac{r}{1+r^2} \leq |G(z)| \leq \frac{r}{1-r^2} \quad (|z| = r < 1)$$

Using the definition of subordination between analytic function, and the equation (3.1), we see that there is an analytic function $w(z)$ with $|w(z)| \leq |z|$ such that

$$\frac{zf'(z)}{G(z)} = \varphi(w(z))$$

or $zf'(z) = G(z)\varphi(w(z))$. Since $w(\mathbb{D}) \subset \mathbb{D}$, we have, by maximum principle for harmonic functions,

$$|f'(z)| \leq \frac{|G(z)|}{|z|} |\varphi(w(z))| \leq \frac{1}{1-r^2} \max_{|z|=r} |\varphi(z)| = \frac{\varphi(r)}{1-r^2}.$$

The other inequality for $|f'(z)|$ is similar. Since the function f is univalent, the inequality for $|f(z)|$ follows from the corresponding inequalities for $|f'(z)|$ by Privalov's Theorem [3, Theorem 7, p. 67].

To prove the sharpness of our results, we consider the functions

$$(3.2) \quad f_0(z) = \int_0^z \frac{\varphi(w)}{1-w^2} dw, \quad f_1(z) = \int_0^z \frac{\varphi(w)}{1+w^2} dw.$$

Define the function g_0 and g_1 by $g_0(z) = z/(1-z)$ and $g_1(z) = z/\sqrt{1+z^2}$. These functions are clearly starlike functions of order $1/2$. Also a calculations shows that

$$-\frac{z^2 f'_k(z)}{g_k(z)g_k(-z)} = \varphi(z) \quad (k = 0, 1).$$

Thus the function f_0 satisfies the subordination (1.1) with g_0 while the function f_1 satisfies it with g_1 ; therefore, these functions belong to the class $\mathcal{K}_s(\varphi)$. It is clear that the upper estimates for $|f'(z)|$ and $|f(z)|$ are sharp for the function f_0 given in (3.2) while the lower estimates are sharp for f_1 given in (3.2). \square

Remark 3.1. We note that Xu et al. [12] also obtained a similar estimates and our results differ from their in the hypothesis. Also we have shown that the results are sharp. Our hypothesis is same as the one assumed by Ma and Minda [6].

Remark 3.2. For the choice $\varphi(z) = (1+z)/(1-z)$, our result reduces to [2, Theorem 3, p. 126] while, for the choice $\varphi(z) = (1+(1-2\gamma)z)/(1-z)$, it reduces to following estimates (obtained in [5, Theorem 4, p. 1151]) for $f \in \mathcal{K}_s(\gamma)$:

$$\frac{1-(1-2\gamma)r}{(1+r)(1+r^2)} \leq |f'(z)| \leq \frac{1+(1-2\gamma)r}{(1-r)(1-r^2)}$$

and

$$(1-\gamma) \ln \frac{1+r}{\sqrt{1+r^2}} + \gamma \arctan r \leq |f(z)| \leq \frac{\gamma}{2} \ln \frac{1+r}{1-r} + (1-\gamma) \frac{r}{1-r}$$

where $|z| = r < 1$. Also our result improves the corresponding results in [11].

Remark 3.3. Let $k := \lim_{r \rightarrow 1-} \int_0^r \varphi(-t)/(1+t^2)dt$. Then the disk $\{w \in \mathbb{C} : |w| \leq k\} \subseteq f(\mathbb{D})$ for every $f \in \mathcal{K}_s(\varphi)$.

4. A SUBORDINATION THEOREM

It is well-known [8] that f is starlike if $(1-t)f(z) \prec f(z)$ for $t \in (0, \delta)$, where δ is a positive real number; also the function is starlike with respect to symmetric points if $(1-t)f(z) + tf(-z) \prec f(z)$. In the following theorem, we extend these results to the class \mathcal{K}_s . The proof of our result is based on the following version of a lemma of Stankiewicz [8].

Lemma 4.1. *Let $F(z, t)$ be analytic in \mathbb{D} for each $t \in (0, \delta)$, $F(z, 0) = f(z)$, $f \in \mathcal{S}$ and $F(0, t) = 0$ for each $t \in (0, \delta)$. Suppose that $F(z, t) \prec f(z)$ and that*

$$\lim_{t \rightarrow 0^+} \frac{F(z, t) - f(z)}{zt^\rho} = F(z)$$

exists for some $\rho > 0$. If F is analytic and $\operatorname{Re}(F(z)) \neq 0$, then

$$\operatorname{Re} \left(\frac{F(z)}{f'(z)} \right) < 0.$$

Theorem 4.1. *Let $f \in \mathcal{S}$ and $g \in \mathcal{S}^*(1/2)$. Let $\delta > 0$ and $f(z) + tg(z)g(-z)/z \prec f(z)$, $t \in (0, \delta)$. Then $f \in \mathcal{K}_s$.*

Proof. Define the function F by $F(z, t) = f(z) + tg(z)g(-z)/z$. Then $F(z, t)$ is analytic for every fixed t and $F(z, 0) = f(z)$ and by our assumption, $f \in \mathcal{S}$. Also

$$\lim_{t \rightarrow 0^+} \frac{F(z, t) - f(z)}{z} = \frac{g(z)g(-z)}{z} := F(z).$$

The function F is analytic in \mathbb{D} (of course, one has to redefine the function F at $z = 0$ where it has removable singularity.) Since all the hypothesis of Lemma 4.1 are satisfied, we have

$$\operatorname{Re} \left(\frac{g(z)g(-z)}{z^2 f'(z)} \right) < 0.$$

Since a function $p(z)$ has negative real part if and only if its reciprocal $1/p(z)$ has negative real part, we have

$$\operatorname{Re} \left(\frac{z^2 f'(z)}{g(z)g(-z)} \right) < 0.$$

Thus, $f \in \mathcal{K}_s$. □

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